Improved bounds for randomly colouring simple hypergraphs

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University of Edinburgh

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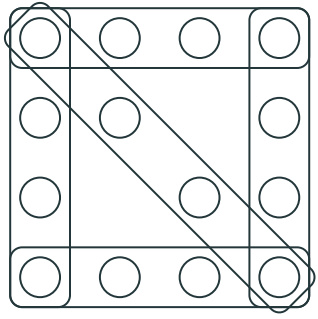
Hypergraph (proper) colouring

Classical combinatorial/computational problem!



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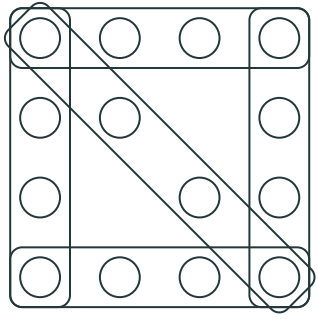

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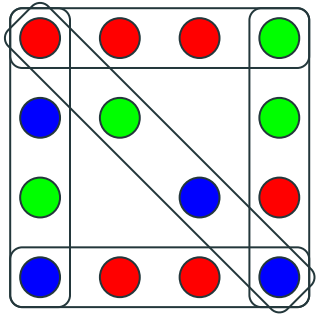
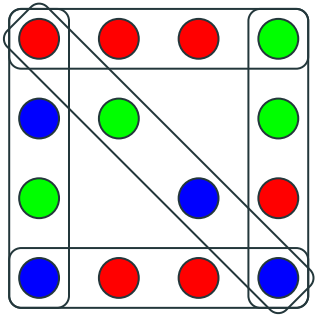
- Hypergraph $(V, \mathcal{E})$
  - Hyperedge $e \in \mathcal{E} : e \subseteq V$
- Proper colouring
  - Forbidding monochromatic hyperedges



Computational problems

Three computational problems:



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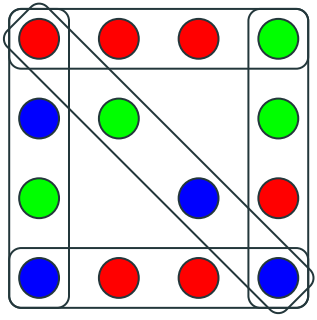

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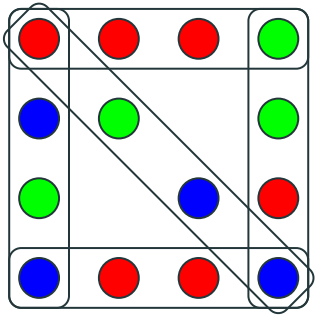

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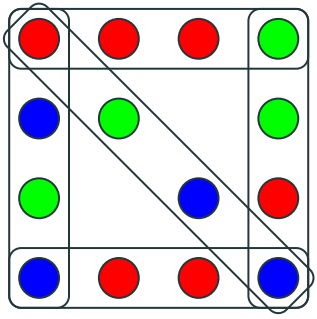

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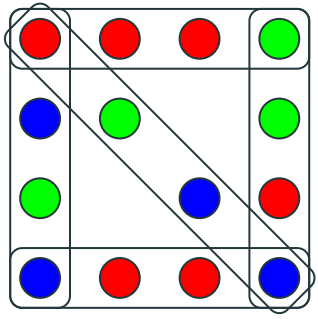

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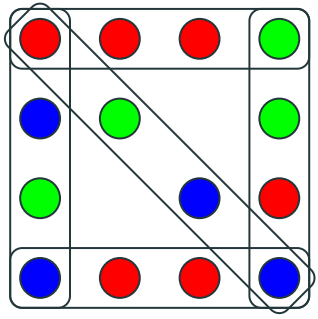

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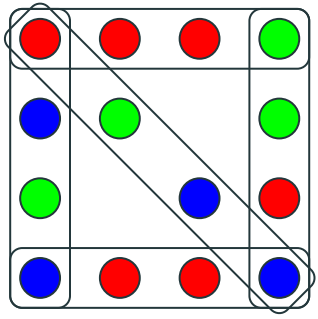

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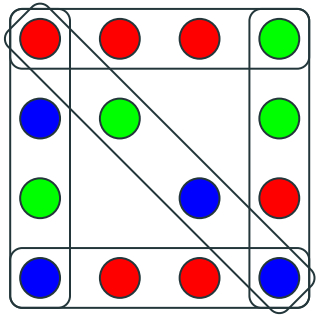

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Posing restrictions to input instances?

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If

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- Each depends on $\leq D$ other events. maximum degree of vertices

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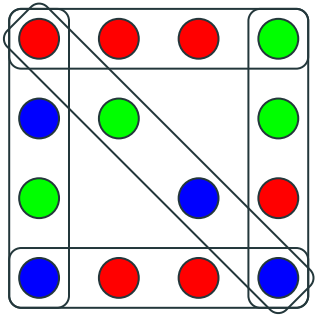

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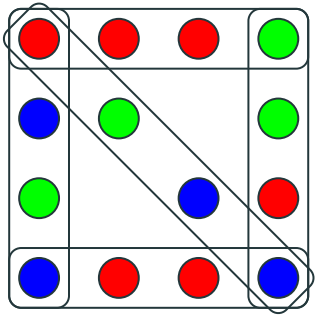

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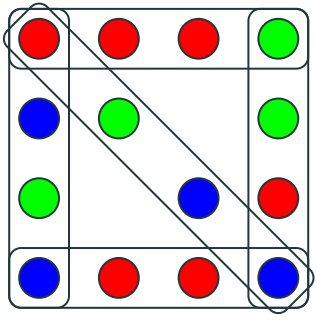

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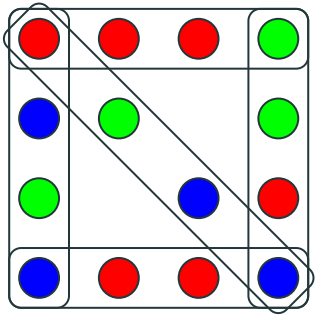

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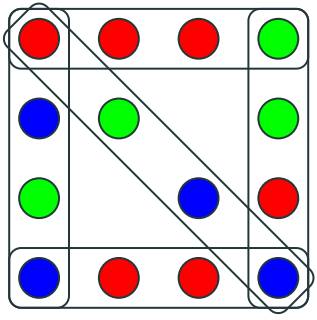

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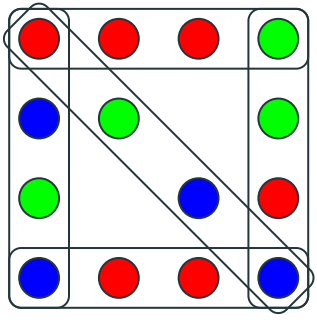

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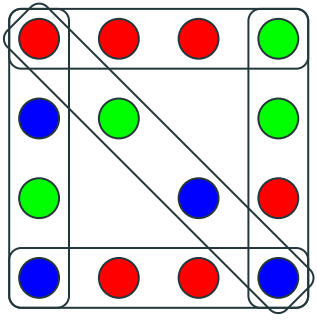

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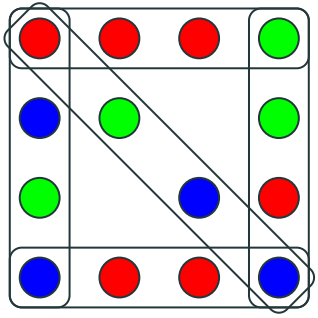
Open problem: computational threshold for sampling problem

Simple (aka. linear) hypergraphs

Simple hypergraph:

- Overlap of two hyperedges ≤ 1



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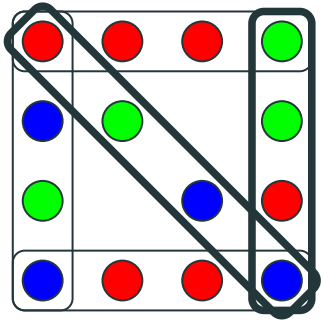

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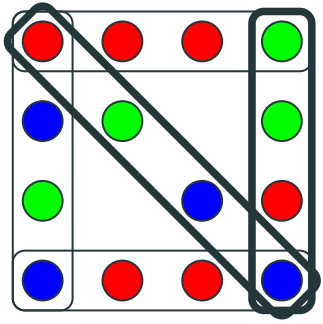

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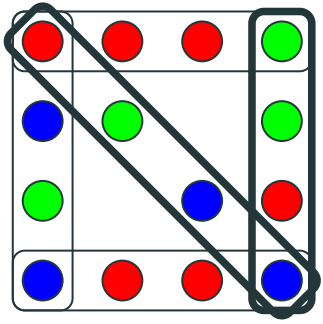

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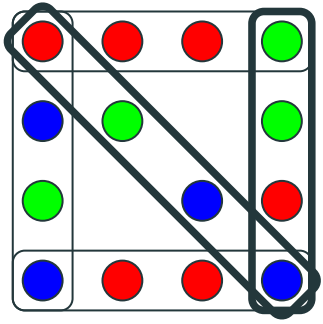

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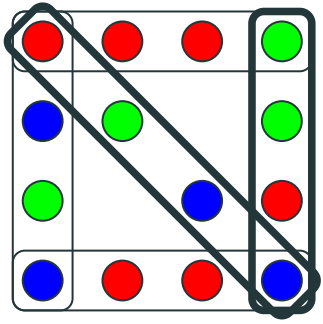

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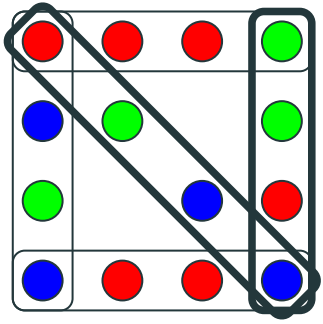

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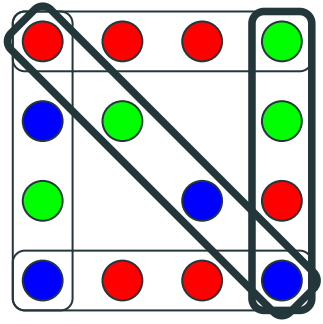

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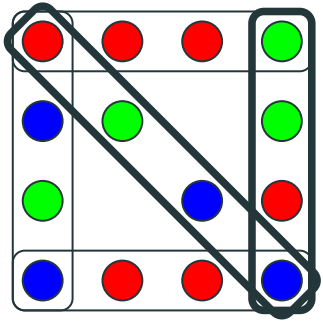

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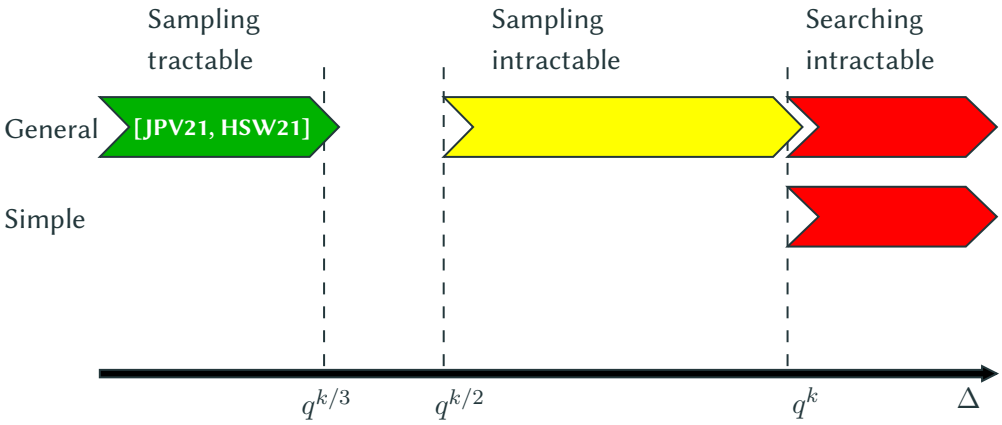
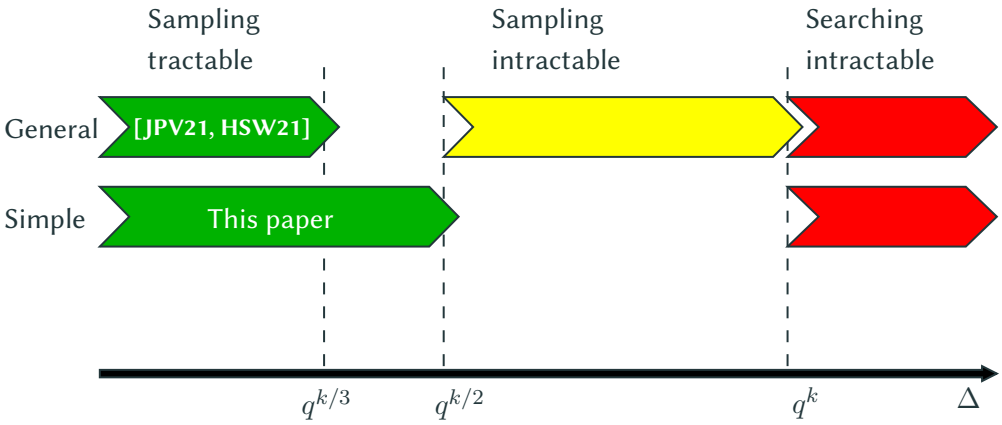
Our result


Sampling
tractable
Sampling
intractable
Searching
intractable
General
[JPV21, HSW21]
Simple
$q^{k/3}$
$q^{k/2}$
q^k
Δ

Our result

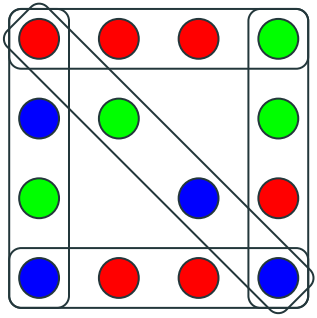


Theorem

There exists an algorithm such that, for any $\delta > 0$, given a k-uniform Δ-degree hypergraph as an input, the algorithm outputs an almost uniform random q-colouring, if $k \geq 20\left(1 + \frac{1}{\delta}\right)$ and $\Delta \leq 0.1^k q^{k/2-(k\delta+1/\delta)}$. The running time is $\tilde{O}(\mathrm{poly}(\Delta k) \cdot n^{1.01})$.

Glauber dynamics

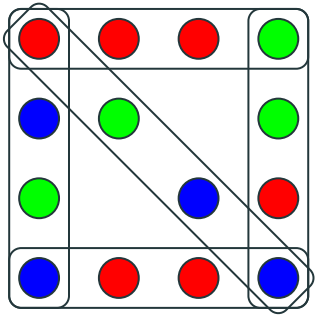
Natural approach: Glauber dynamics



Glauber dynamics

Natural approach: Glauber dynamics

- Uniformly choose one vertex
- Update its value according to its marginal



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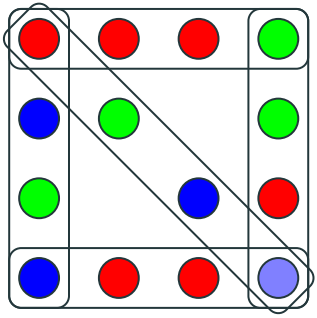

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- Uniformly choose one vertex
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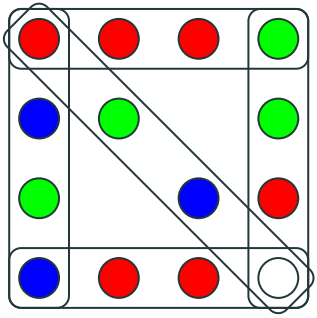
$$\Pr[\bullet] = \frac{1}{3}, \Pr[\bullet] = \frac{1}{3}, \Pr[\bullet] = \frac{1}{3}$$



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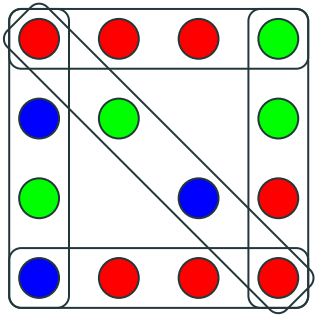

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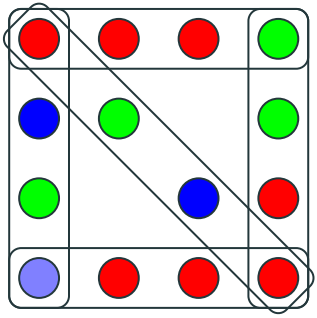

Glauber dynamics

Natural approach: Glauber dynamics

- Uniformly choose one vertex
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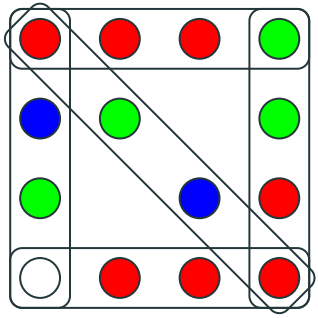
$$\Pr[\bullet] = 0, \Pr[\bullet] = \frac{1}{2}, \Pr[\bullet] = \frac{1}{2}$$



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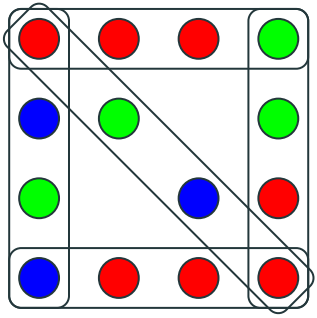

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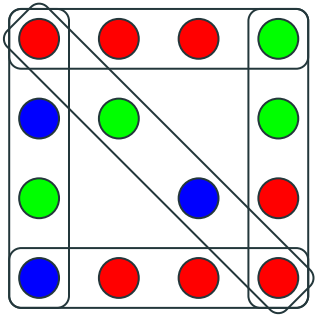

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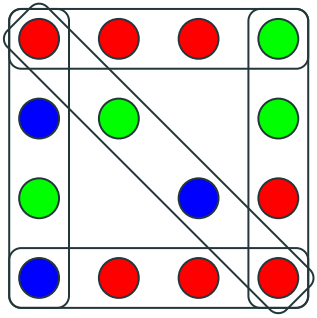

Stationary distribution is uniform (the correct distribution).

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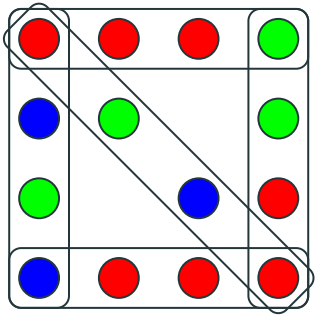
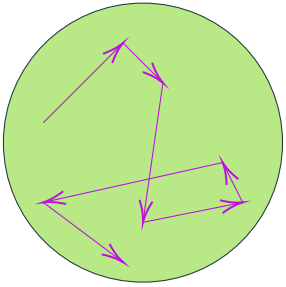


Stationary distribution is uniform (the correct distribution).

Does this chain *mix rapidly* (i.e., converge to stationary distribution quickly)?

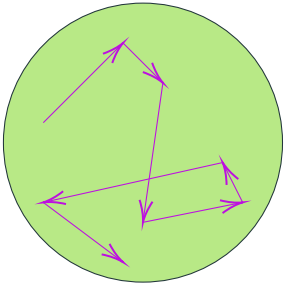
Dystopia of MCMC?



Well connected

Fast mixing

Dystopia of MCMC?



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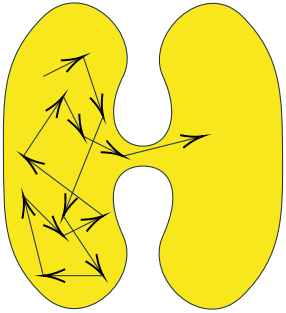

Poorly connected
Slow mixing

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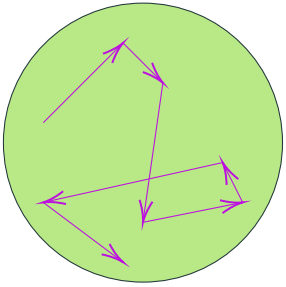

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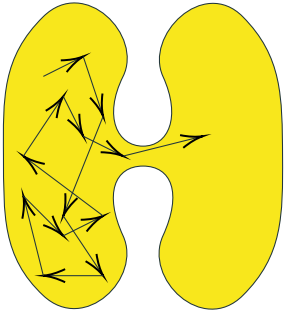

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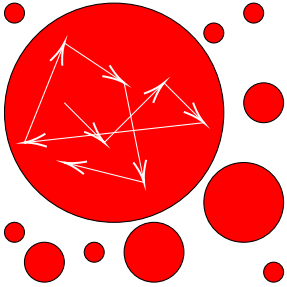
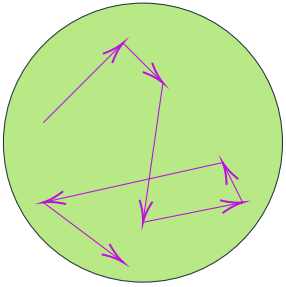


Not connected
Not mixing

Dystopia of MCMC?



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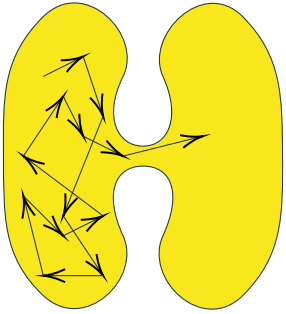

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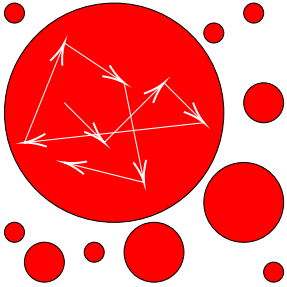

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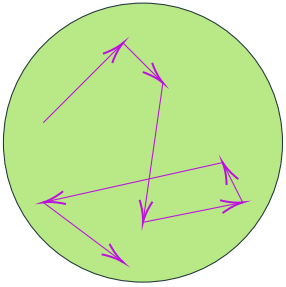

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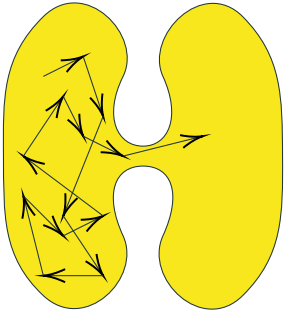

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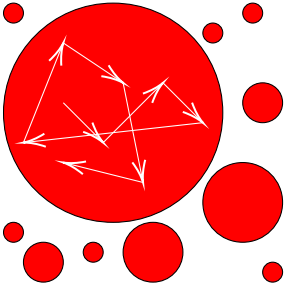

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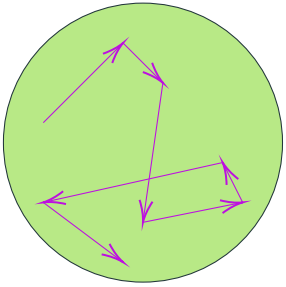
Not mixing

Not that bad if there is a giant component — start from a random configuration

Dystopia of MCMC?



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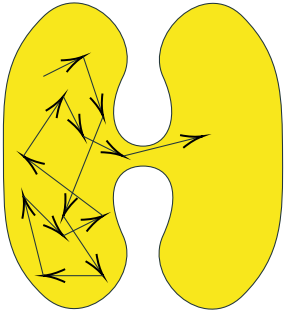

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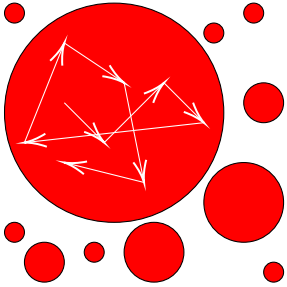

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- Simple hypergraph with $q \geq \max\{\Theta_k(\log n), \Theta_k(\Delta^{\frac{1}{k-1}})\}$ **[Frieze-Anastos'17]**

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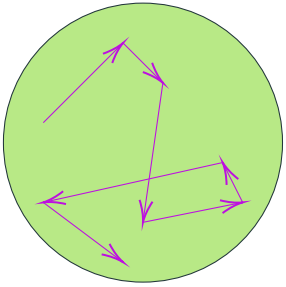

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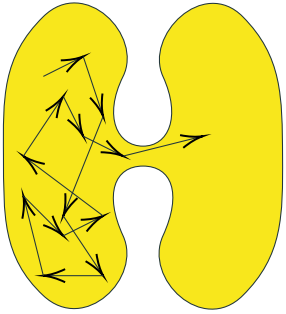

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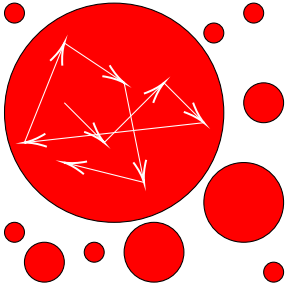

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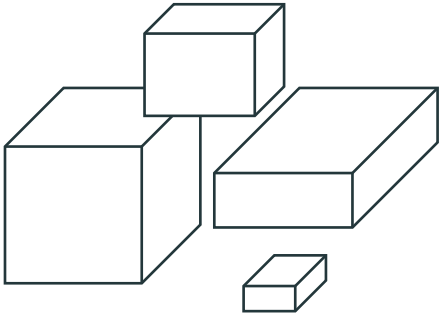
- Constant number of colours?

Markov chain projection [Feng-Guo-Yin-Zhang'21]



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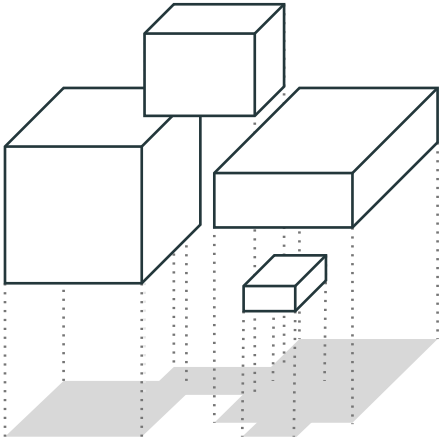

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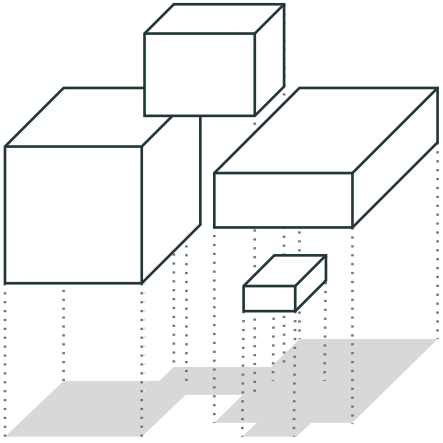

- Disconnected / poorly connected $\xrightarrow{\text{projection}}$ well connected

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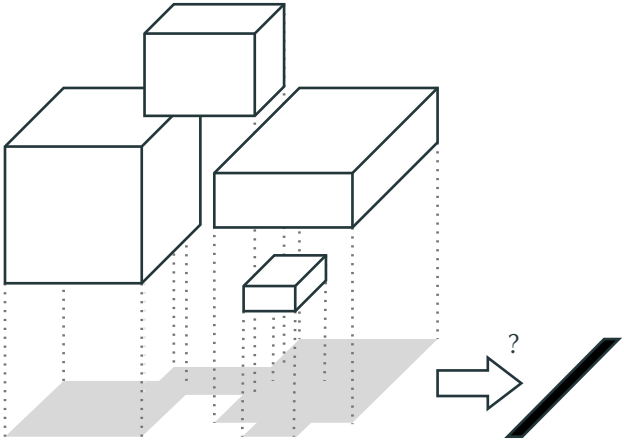

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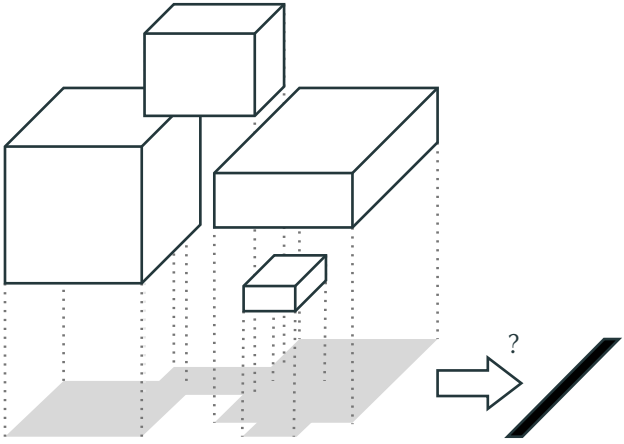

- Disconnected / poorly connected $\xrightarrow{\text{projection}}$ well connected
- Harder to simulate transition / recover a sample

The algorithm

Algorithm:

- Run Glauber dynamics on the projected distribution to get a projected sample $\boldsymbol{Y}$

The algorithm

Algorithm:

- Run Glauber dynamics on the projected distribution to get a projected sample $\boldsymbol{Y}$
- Sample a proper colouring $\boldsymbol{X}$ conditioned on its projection being $\boldsymbol{Y}$

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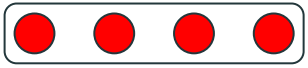
- Simulate transition and get the final sample

Bucketing [Feng-He-Yin'21]

- Disconnection arises from hard constraints:



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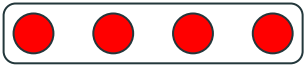
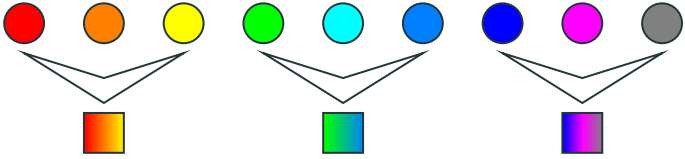


- Projection by bucketing:



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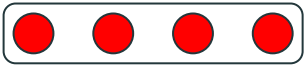

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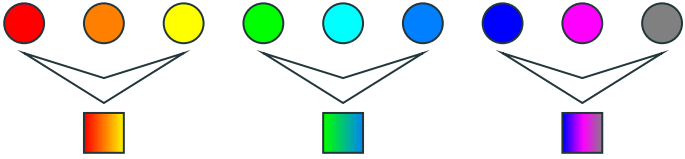

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$$\Pr\left[\ \right] > 0$$

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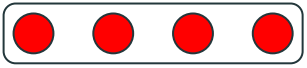

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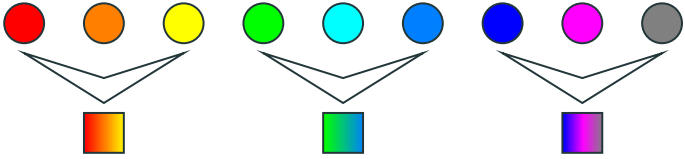

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$$\Pr\Big[\ \Big] > 0$$

despite that

$$\Pr\Big[\ \Big] < \Pr\Big[\ \Big] = \Pr\Big[\ \Big]$$

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- Choose a proper projection
  - Bucketing
- Analyse the mixing time of projected chain
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Current $\Delta \lesssim q^{k/3}$ barrier: trade-off between mixing and implementation.

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Improve both to get $\Delta \lesssim q^{k/2}$ on simple hypergraphs.

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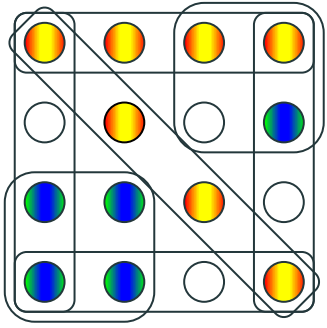
- We focus on implementation in this talk.

Fast implementation



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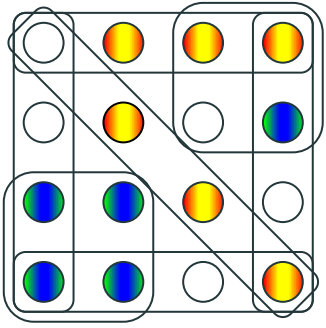

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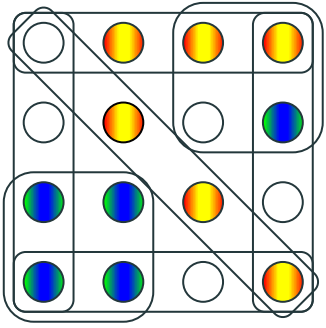

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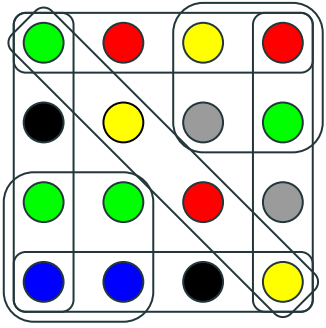

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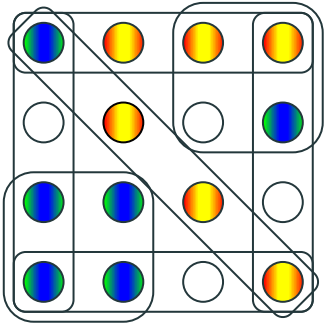

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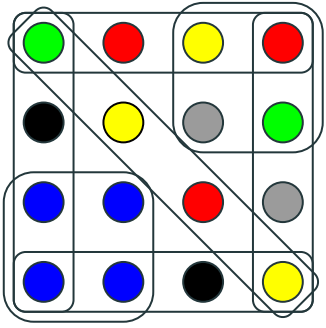

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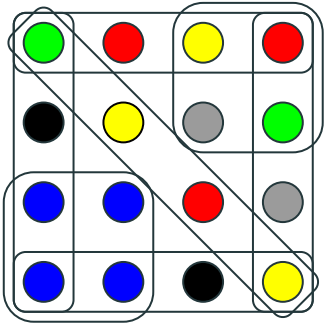

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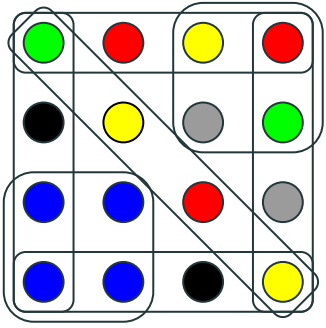

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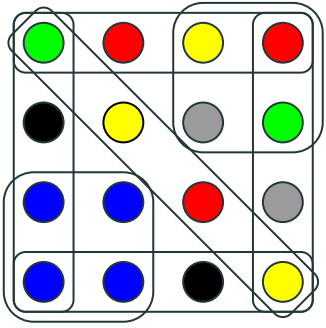

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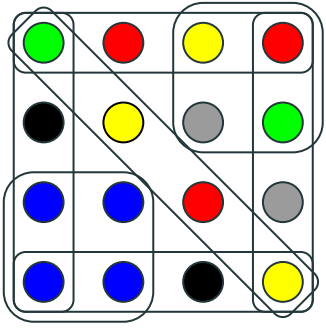

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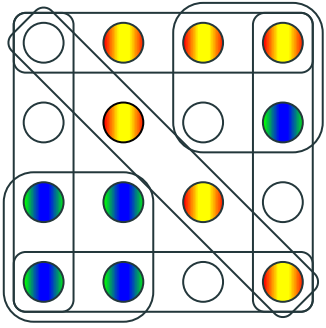

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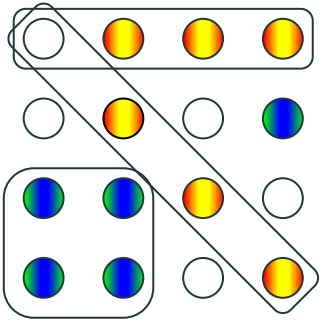

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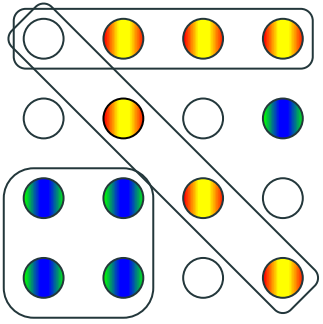

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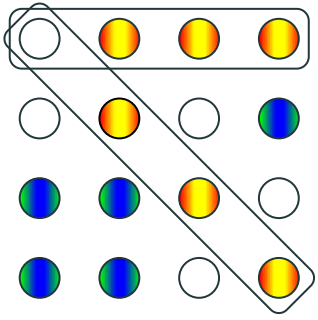

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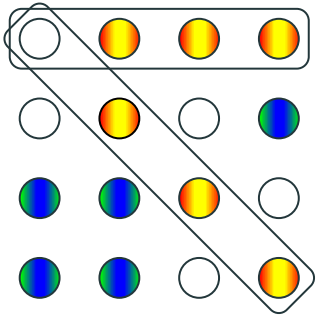
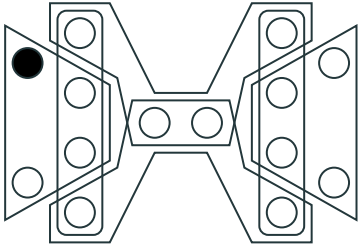
- We are done if number of hyperedges is $O(\log n)$ w.h.p.



Connected component

Union bound over all possible size-α(#edges) components:



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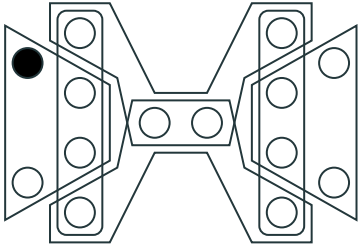

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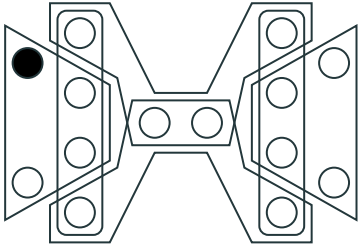

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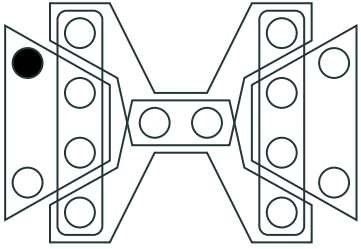

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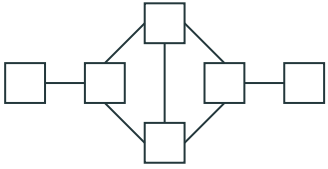

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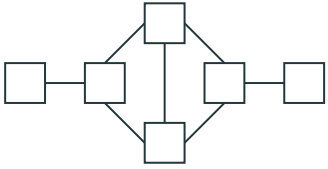

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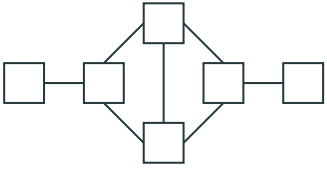

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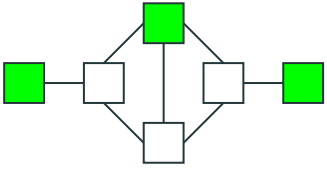

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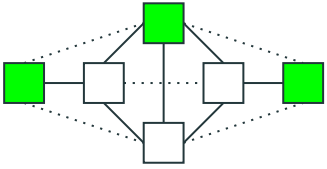

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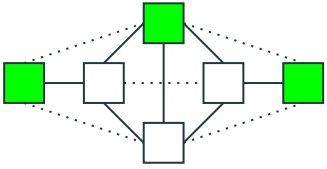

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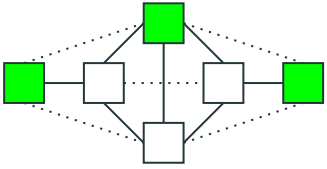
- Union bound over all 2-trees instead.



Do 2-trees suffice?

Assuming bucketing into $\sqrt{q}$ buckets.

$$\sum_{\ell} \Pr[\text{size-}\ell \text{ 2-tree exists}] < 1.$$

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Lemma (Corollary of [Borgs-Chayes-Kahn-Lovász'13])

Let G be a graph with maximum degree D and v is a vertex. Then the number of 2-trees in G of size ℓ containing v is at most $(eD^2)^{\ell-1}/2$.

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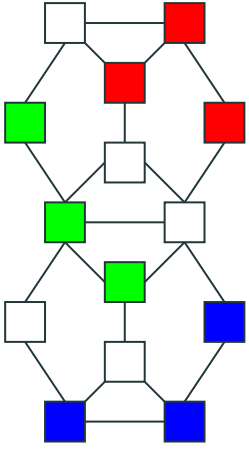
2-block-trees

Idea: utilising small overlaps!

- Single vertex in 2-tree $\rightarrow$ size-θ component (block)



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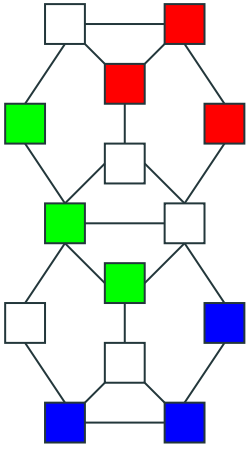

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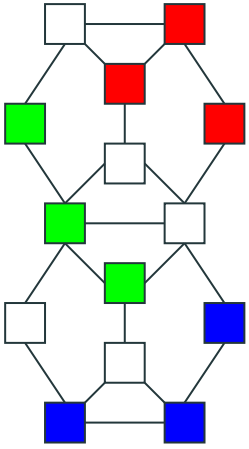

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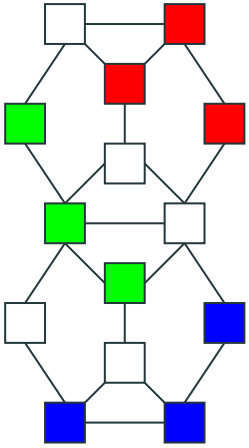

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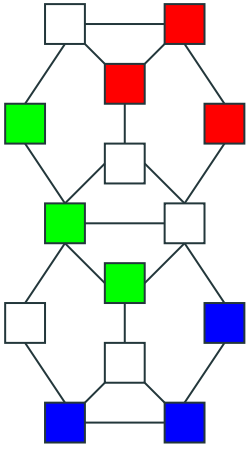

Requires:

$$\Delta \lesssim q^{\frac{k}{2+O(1/\theta)}}$$

Future directions

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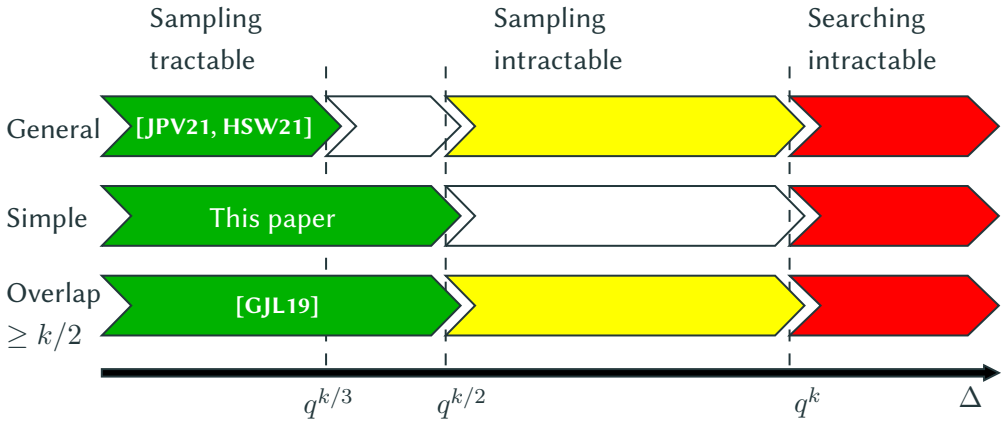
  - Better condition?
- Utilising overlap information?
  - Partial rejection sampling **[Guo-Jerrum-Liu'19]** gives transition at $\Delta \approx q^{k/2}$ when overlaps are large.


Sampling tractable
Sampling intractable
Searching intractable
General
[JPV21, HSW21]
Simple
This paper
Overlap $\geq k/2$
[GJL19]
$q^{k/3}$
$q^{k/2}$
q^k
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Thank you!

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