Inapproximability of counting hypergraph colourings

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July 31, 2021

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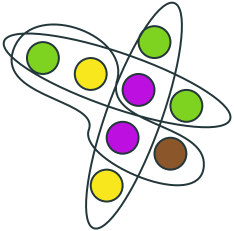
- K-uniform: K vertices in each hyperedge;
- Δ-degree: each vertex appears in $\leqslant \Delta$ hyperedges;
- Event: a hyperedge is monochromatic;
- $p = 1/q^{K-1}$, $D = K\Delta - 1$;
- LLL condition: $\Delta \leqslant \frac{q^{K-1}}{eK}$



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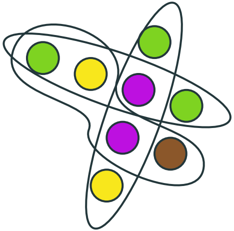

Other kinds of LLL-type problems:

- Boolean K-SAT;
- Constraint Satisfaction Problem;
- ...

Algorithmic LLL and Sampling/Counting LLL

Proof of LLL is non-constructive!

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A lot of progress from algorithmic side! Recent result by **[HSW21]** (even perfect samplers):

LLL-type problem	Algorithmic bound	LLL condition
Hypergraph Colourings	$\Delta \lesssim q^{K/3}$	$\Delta \lesssim q^{K}$
Boolean K-SAT	$\Delta \lesssim 2^{0.175K}$	$\Delta \lesssim 2^{K}$
General Atomic CSPs	$p^{0.175}\Delta \lesssim 1$	$p\Delta \lesssim 1$

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Intractability region of sampling / counting *vs.* LLL? $\Leftarrow$ **Main topic of the work.**

Is hardness transition the same for counting and searching?

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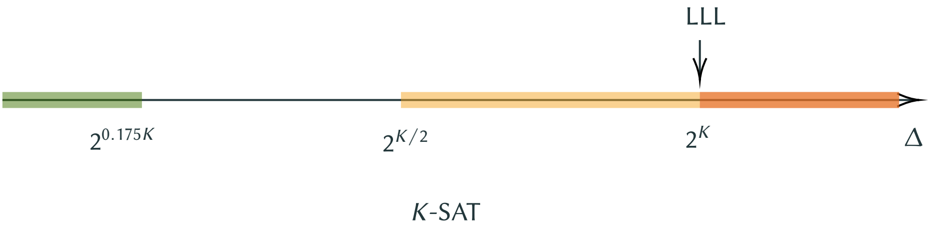

Theorem ([BGGGS16])

If $\Delta \gtrsim 2^{K/2}$, then it is **NP***-hard to sample a satisfying assignment from K-CNF with variable degree $\leqslant \Delta$, even when there is no negation in the formula (aka monotone).*

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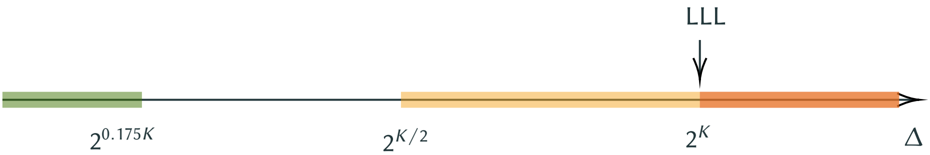

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Not true for K-SAT!

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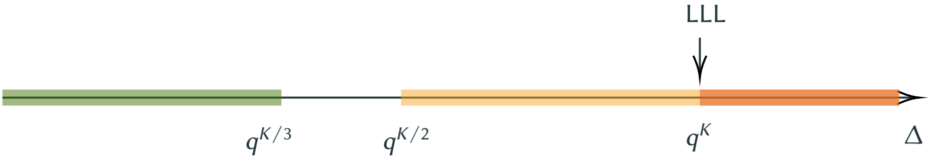
Hardness for searching takes place near LLL indeed, again ...

Theorem

Let $q, K \geqslant 2$ be integers with $(q, K) \neq (2, 2)$. It is **NP***-hard to find a q-colouring on K-uniform simple hypergraphs of maximal degree at most Δ, when $\Delta \geqslant 2Kq^K \ln q + 2q$.*

Our results



Hypergraph Colouring

Algorithmic bound closer to LLL ... Chance for hardness transition to coincide at LLL??

Hardness for searching takes place near LLL indeed, again ...

... but searching and counting do not coincide either! (at least for even q)

Theorem

Let $q \geqslant 4$ be even, $K \geqslant 4$ be even, and $\Delta \geqslant 5q^{K/2}$. It is **NP***-hard to approximate the number of proper q-colourings in n-vertex K-uniform hypergraphs of maximum degree at most Δ, even within a factor of 2^{cn} for some constant $c(q, K) > 0$.*

Reduction

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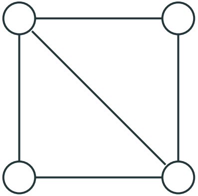

Examples:

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$\boldsymbol{B}$	$\begin{bmatrix} b & 1 & \cdots & 1 \\ 1 & b & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & b \end{bmatrix}$			

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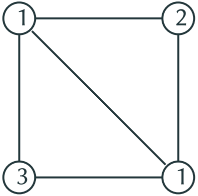

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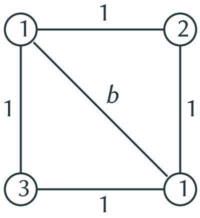

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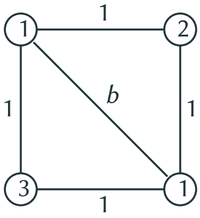

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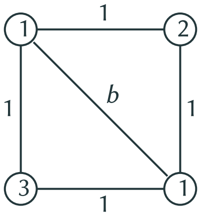

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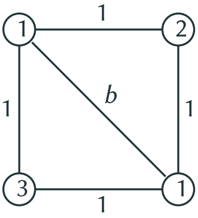

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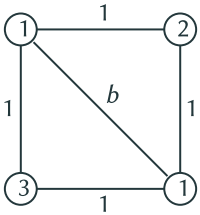

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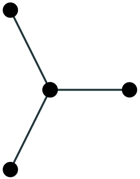
Reduction for counting hypergraph colourings



- Consider Δ-regular graph G.

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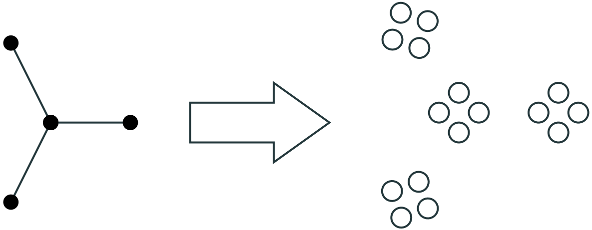

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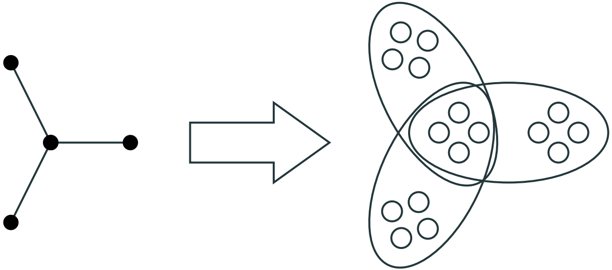

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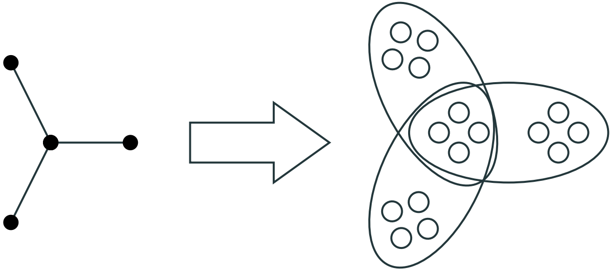

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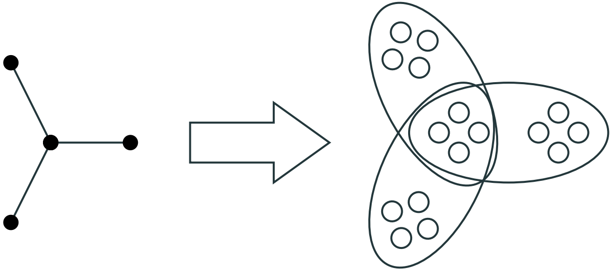

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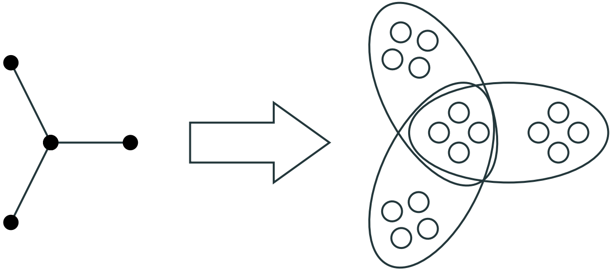

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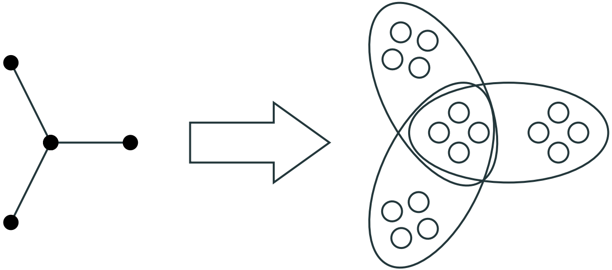

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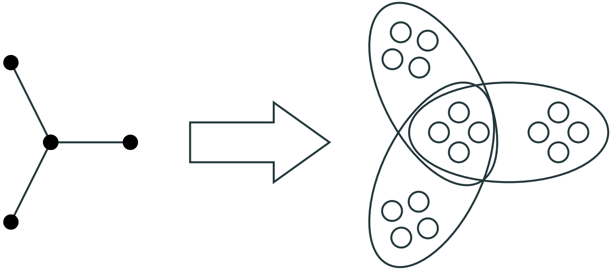

$$B = \begin{bmatrix} t^2 & t & t & \cdots & t \\ t & 0 & 1 & \cdots & 1 \\ t & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & 1 & 1 & \cdots & 0 \end{bmatrix}$$

$$t = (q^k - q)^{1/\Delta}.$$

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- H_G is Δ-regular and $K = 2k$-uniform.
- Consider a proper q-colouring on H_G:
  - H_v has the same colour $s \in [q] \to$ assign v with spin s.
  - H_v has mixed colours $\to$ assign v with spin 0.
- It turns out $Z_{\boldsymbol{B}}(G) = \#\textsc{HyperCol}(H_G)$.

Inapproximability of spin systems

History

[DFJ02]: Hardness of approximating Hard-core model with $\lambda = 1$ (i.e., #IND), $\Delta \geqslant 25$.

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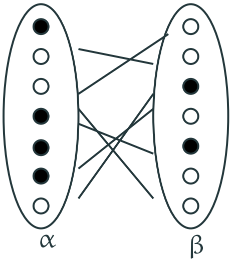
- Gadget: Random (d-regular) bipartite graph $G \sim \mathcal{G}_{n,n,d}$.
- Observe: $d <$ some threshold $\to$ inbalanced "phase".



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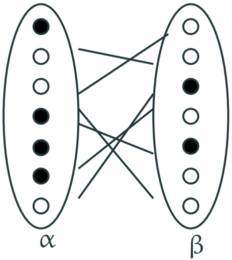

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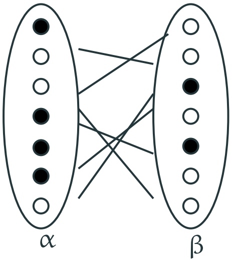

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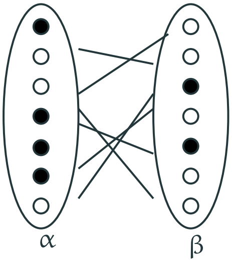

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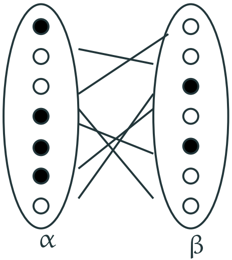

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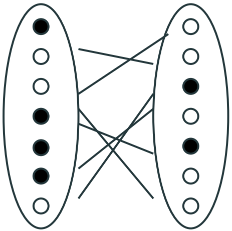

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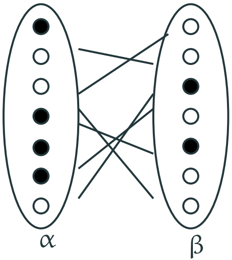

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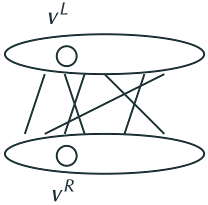

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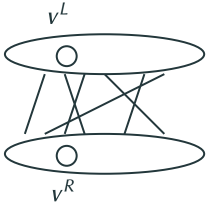

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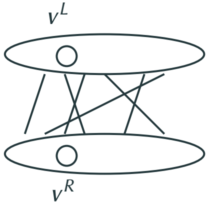

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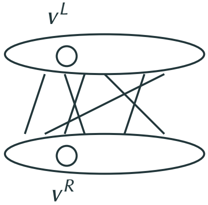

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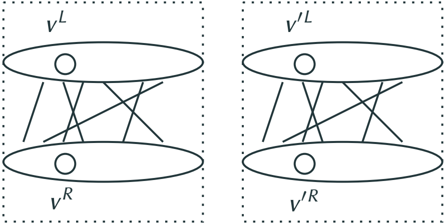

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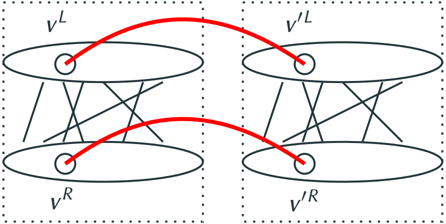

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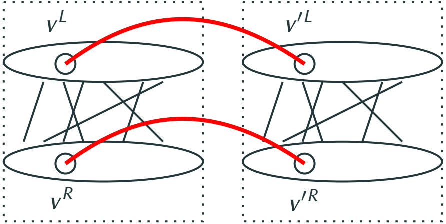

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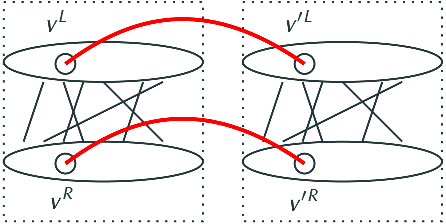

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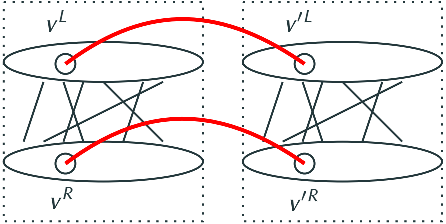

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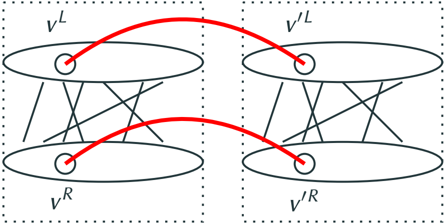

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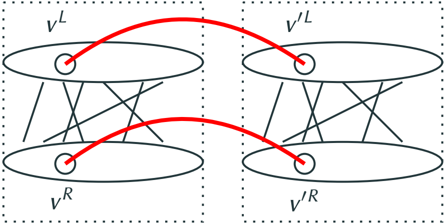

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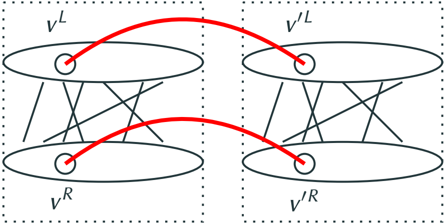

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- $p^+ \neq p^- \implies (1-(p^+)^2)(1-(p^-)^2) < (1-p^+p^-)^2$. Neighbour phases prefer to differ.

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Permutation symmetric: dominant phases can be obtained from each other, by permutating spins while leaving $\boldsymbol{B}$ invariant, or switch $\boldsymbol{\alpha}, \boldsymbol{\beta}$, or both.

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Hessian: the Hessian of $\boldsymbol{\Psi}_1$ is negative-definite.

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Critical points of Φ satisifies ($d := \Delta - 1$):

$$R_i \propto \left(\sum_{j \in [q]} \boldsymbol{B}_{ij} C_j \right)^d ; \qquad C_j \propto \left(\sum_{i \in [q]} \boldsymbol{B}_{ij} R_i \right)^d .$$

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Jacobian stable fixpoints of tree recursion $\iff$ Hessian local maxima of $\mathbf{\Psi}_1$.

Dominant phase analysis

i.e., the proof

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The reason why they can only deal with even q.

Our case

Recall:

Proper q-colouring	Our case
$\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$	$\begin{bmatrix} t^2 & t & t & \cdots & t \\ t & 0 & 1 & \cdots & 1 \\ t & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & 1 & 1 & \cdots & 0 \end{bmatrix}$

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- In fact, **[GŠV15]** considers Potts with $b < \frac{\Delta - q}{\Delta}$.

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- $(q/2, q/2, 0)$ with $R_0/R_1 = C_0/C_3$.
- $q = 6, k = 3$:

$$\mathbf{r} = 0.9863, 0.0045, 0.0045, 0.0045, 0.0001, 0.0001, 0.0001;$$

$$\mathbf{c} = 0.9863, 0.0001, 0.0001, 0.0001, 0.0045, 0.0045, 0.0045.$$

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- $(q/2, q/2, 0)$ with $R_0/R_1 = C_0/C_3$.
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Use a more careful interpolate-and-perturb argument to show 2-supported is global maxima.

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- $(q, 0, 0)$ with $R_0/R_1 \neq C_0/C_1$ can be regarded as a limit of $(q_1, q_2, 0)$ fixpoint.

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- Lies in uniqueness region.
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- Translation-invariant.

$$\mathbf{r} = \mathbf{c} = 0.984, 0.003, 0.003, 0.003, 0.003, 0.003, 0.003.$$

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Dominant phase satisfies $\alpha = \beta$. Cannot apply **[GŠV15]**.

Summary

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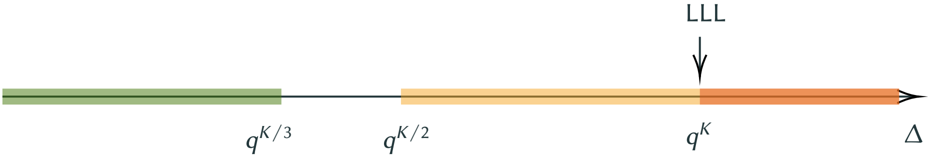
Algorithmic side: close the gap between $q^{K/3}$ and $q^{K/2}$.

- Which one is the computational transition threshold? (We guess 1/2.)



Hypergraph Colouring

Thank you!

arXiv: 2107.05486