On the Degree of Boolean Functions as Polynomials over $\mathbb{Z}_m$

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Currently best upper bound of modular counting circuits: $\mathbf{ACC}^0 \not\supseteq \mathbf{NEXP}$, which builds on Williams' breakthrough algorithmic method for circuit lower bounds **[Williams, 2011]**.

Polynomial Representation and Degree

Represent every Boolean function $f : \{0,1\}^n \to \{0,1\}$ by polynomial:

$$\sum_{a\in\{0,1\}^n} f(a) \left(\prod_{i:a_i=1} x_i\right) \left(\prod_{i:a_i=0} (1-x_i)\right) =: \sum_{S\subseteq[n]} c_S \prod_{i\in S} x_i.$$

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A function is *non-degenerated*, if it depends on all n input bits.

Theorem ([Gopalan, Lovett and Shpilka, 2009])

For all non-degenerated $f : \{0,1\}^n \to \{0,1\}$ *and different primes* p, q,

$$\deg_q(f) \geq \frac{n}{\lceil \log_2 p \rceil \deg_p(f) p^{2\deg_p(f)}}.$$

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$\deg_{pq}(f)$ vs $\deg(f)$

Conjecture

For any Boolean function f,

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Best separation so far is quadratic **[Li and Sun, 2017]**.

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This conjecture is true for symmetric functions **[Lee et al., 2015]**.

Our Results

Theorem ([Li and Sun, 2017])

For any positive integer m with at least two different prime factors p, q and any non-trivial symmetric function $f : \{0,1\}^n \to \{0,1\}$, we have

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For any positive integer m *with at least two different prime factors* p, q *and any non-trivial symmetric function* $f : \{0,1\}^n \to \{0,1\}$*, we have*

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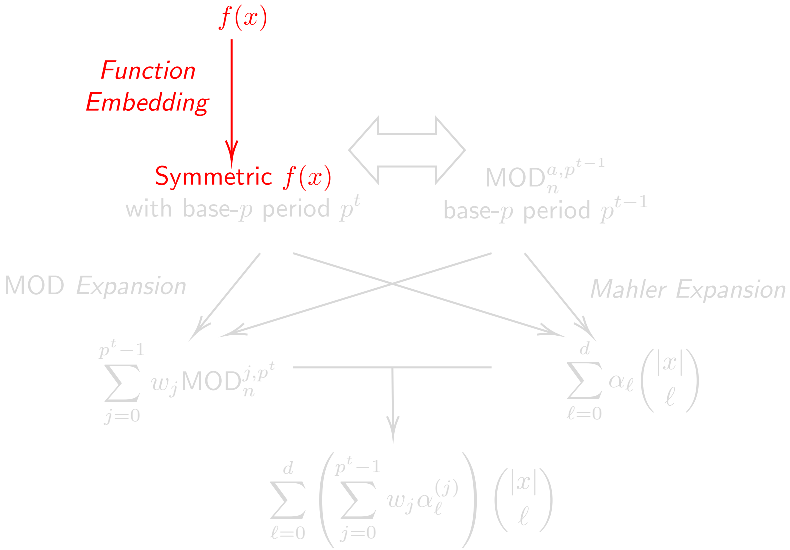

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Symmetric Function Embedding

Lemma

Let $f : \{0,1\}^n \to \{0,1\}$ be a non-degenerate Boolean function. Then there exists a set of indices $S \subseteq [n]$ with $|S| = \omega(1)$, and a restriction $\sigma : [n] \setminus S \to \{0,1\}$ such that $f|_\sigma$ is a non-trivial symmetric Boolean function.

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Proved by hypergraph Ramsey theory.

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Suppose $M(f)$ is a complexity measure. If M is non-increasing w.r.t. restrictions (i.e., $M(f) \geq M(f|_\sigma)$), then

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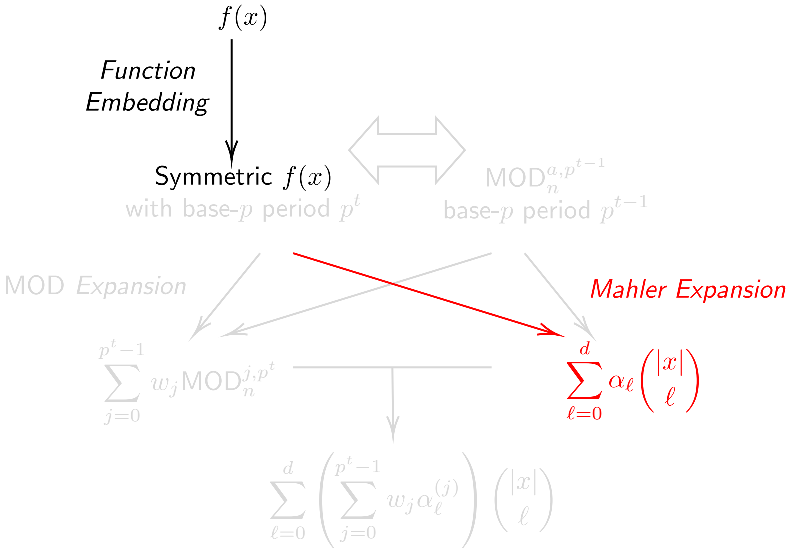
$r(n) \approx \sqrt{\log^*(n)}$ grows extremely slow, but suffices for our purpose.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Theorem (Mahler expansion)

Assume that $f : \{0,1\}^n \to \{0,1\}$ is a symmetric Boolean function, and F is the corresponding univariate version. Let $d := \max\{n, m-1\}$. Then there exists a unique sequence $\alpha_0, \alpha_1, \cdots, \alpha_d \in \mathbb{Z}_m$ such that

$$\sum_{j=0}^{d} \alpha_j \binom{t}{j} = \begin{cases} F(t), & 0 \le t \le n; \\ 0, & n < m-1 \text{ and } n < t < m. \end{cases}$$

We call $\sum_{j=0}^{d} \alpha_j \binom{t}{j}$ the *Mahler expansion* of F over $\mathbb{Z}_m$, and α_j the j-th *Mahler coefficient*.

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Let $n = 2$ and $f(x) = x_0 \vee x_1$. On $\mathbb{Z}_5$, its Mahler expansion is

$$F(x) = \binom{|x|}{1} + 4\binom{|x|}{2} + 2\binom{|x|}{4}.$$

But $\deg_5(f) = 2$.

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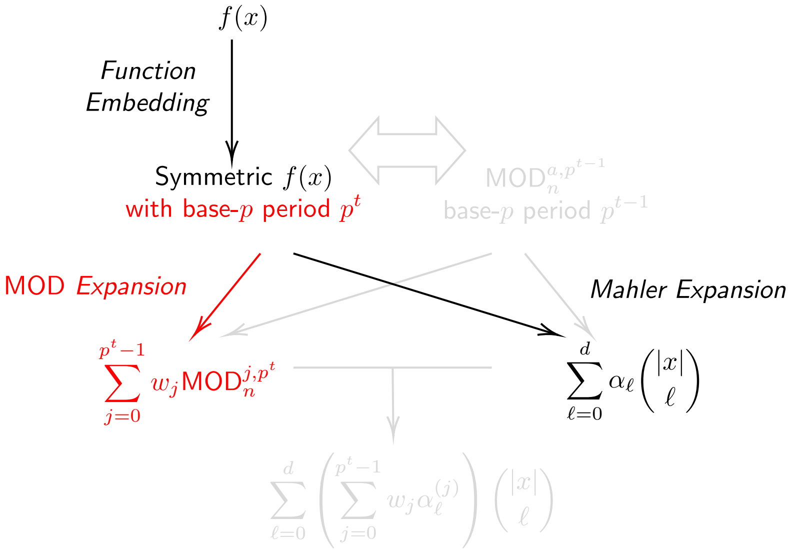
Fact

$\deg_m(f) = \max\{\ell : \alpha_\ell \not\equiv 0 \pmod{m}, \ell \leq n\}$.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Every m-periodic function can be spanned by $\{\mathsf{MOD}_n^{a,m}(x)\}_{a=0}^{m-1}$.

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- Example: The not-all-equal NAE function is defined as $\mathsf{NAE}_n(x_1, \ldots, x_n) := \mathbb{I}[\exists i, j \text{ s.t. } x_i \neq x_j]$. Then $\pi_3(\mathsf{NAE}_3) = 3$ while $\pi_3(\mathsf{NAE}_4) = 9$.

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Theorem ([Wilson, 2006])

For prime p and positive integers t, k, denote $d := (k-1) \cdot \varphi(p^t) + p^t - 1$. Then for any $n \geq d$, we have $\deg_{p^k}(\mathsf{MOD}_n^{0,p^t}) = d$.

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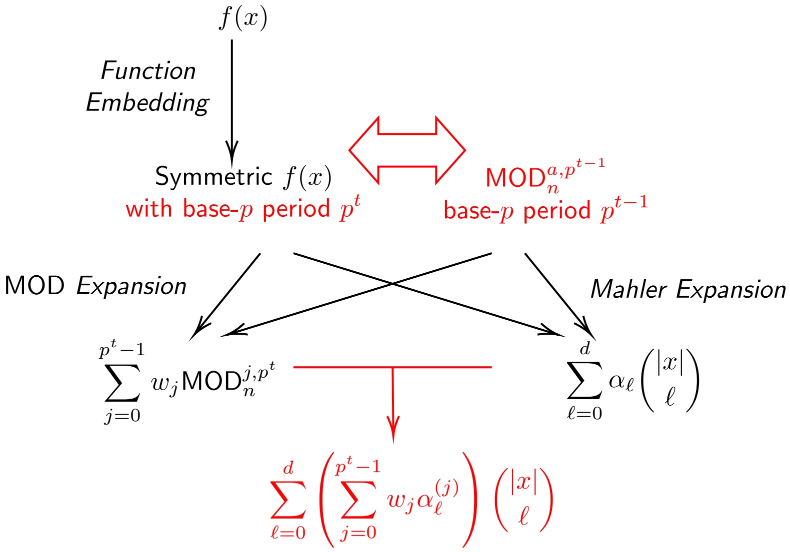
Corollary

For prime p and positive integers t, k, denote $d := (k-1) \cdot \varphi(p^t) + p^t - 1$. Then for any $n \geq d$ and a, we have $\deg_{p^k}(\mathsf{MOD}_n^{a,p^t}) = d$.


$f(x)$
Function Embedding
Symmetric $f(x)$
with base-p period p^t
$\mathsf{MOD}_n^{a,p^{t-1}}$
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MOD *Expansion*
Mahler Expansion
$\sum_{j=0}^{p^t-1} w_j \mathsf{MOD}_n^{j,p^t}$
$\sum_{\ell=0}^{d} \alpha_\ell \binom{|x|}{\ell}$
$\sum_{\ell=0}^{d} \left(\sum_{j=0}^{p^t-1} w_j \alpha_\ell^{(j)} \right) \binom{|x|}{\ell}$

Combination of Different Expansions

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- So $\boldsymbol{S}\boldsymbol{w} \neq 0$, implying a high-order Mahler coefficient of f.

From Primes to Their Product

Theorem

For any prime p, positive integer k, and non-trivial symmetric function $f : \{0,1\}^n \to \{0,1\}$ with sufficiently large n,

$$\deg_{p^k}(f) \geq (p-1) \cdot k.$$

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Lemma

For any prime p and non-trivial symmetric function $f : \{0,1\}^n \to \{0,1\}$,

$$\deg_p(f) \geq \min\left\{\frac{n}{2}, \left(1 - \frac{1}{p}\right)\pi_p(f)\right\}.$$

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Let g be an a-periodic and b-periodic function on domain $\{0,1,\ldots,n\}$ with $gcd(a,b) = 1$ and $n \geq a + b - 2$. Then g is a constant function.

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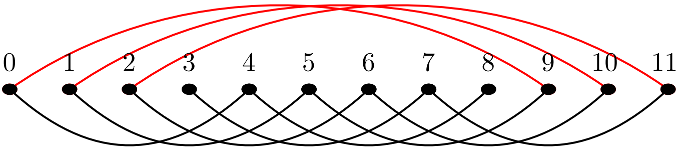

$a = 4$, $b = 9$ and $n = a + b - 2$

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Combine both to get $\deg_{pq}(f) > \frac{n+2}{2+\frac{1}{p-1}+\frac{1}{q-1}} > \frac{n}{2+\frac{1}{p-1}+\frac{1}{q-1}}$.

Instance with Factor $1/2$

Lemma

If $1, a_1, \cdots, a_k$ are linearly independent over $\mathbb{Q}$, then for any $\varepsilon > 0$, there exist infinitely many $\ell \in \mathbb{N}_+$ such that $\ell a_i \bmod 1 \in (1-\varepsilon, 1)$ for all i.

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Therefore, $p_i^{r_i}/q^\ell \in (1, p_i^\varepsilon)$ where $r_i = \lceil \ell \cdot \log q / \log p_i \rceil$.

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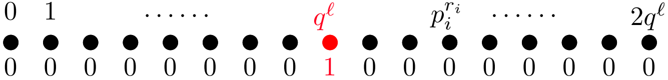

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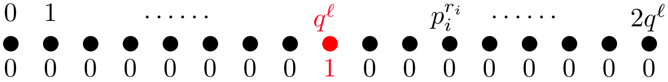

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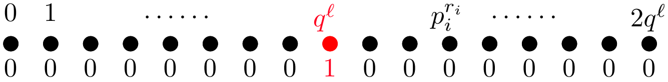

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Finally,

$$\deg_m(f) \overset{\text{CRT}}{=} \max\{\deg_{p_i}(f)\} \le \max\{p_i^{r_i}\} \le \frac{n}{2} \max\{p_i^{\varepsilon}\}.$$

Then let $\varepsilon \to 0$.

Concluding Remarks

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- A related conjecture: $\deg(f) \geq n - O(1)$ for all non-trivial symmetric Boolean functions. **[Gathen and Roche, 1997]**
  - Best lower bound: $\deg(f) \geq n - O(n^{0.525})$.
  - Best instance: $\deg(f) = n - 3$.

Thank you!